

Midterm Test
discreteG, 0405, T1

Mandatory Problems, Part I, You have no choice!

1. Determine the number of nonnegative integer solutions (x, y, z) to $x + y + z = 15$. Determine the number of these solutions satisfying $x \geq 3$, $y \geq 4$ and $z \geq 1$.

Solution. This is a coins-to-children problem. We have 15 coins and three children: x , y and z . In this event, $n = 15$ and $r = 3$ so we have

$$\binom{n+r-1}{r-1} = \binom{17}{2} = 136$$

solutions.

To solve the other part, preallocate seven coins so x gets 4, y gets 3, and z gets 1; 7 coins remain. There is one way to do this. Now allocate the remaining 7; $n = 7$ and $r = 3$ so we have

$$\binom{9}{2} = 36$$

ways to do this.

2. *Comment on the validity of the solution to the following problem.* A committee of 10 people is to be chosen from 8 men and 6 women so that the committee has at least 4 women. Cooley says it can be done $\binom{6}{4}\binom{10}{6}$ ways. She argues that you must first select 4 women from the 6 and you can then select 6 more members freely from the remaining people to fill out the committee. She claims the counting will include all committees of 4, 5, or 6 women and none with fewer.

Solution. If I were a drudge-artist, I'd have to say the right way to solve this one is to divide into cases: exactly 4 women, exactly 5 women, exactly 6 women. To choose exactly 4 women we choose 4 women of the 6 and 6 men of the 8 to get

$$\binom{8}{4} = 70$$

ways. If we choose exactly 5 women, we can do this

$$\binom{8}{5}\binom{6}{5} = 56 \cdot 6 = 336$$

ways. If we choose exactly 6 women we have

$$\binom{8}{6}\binom{6}{4} = 28 \cdot 15 = 420.$$

ways. This gives us a total of 826 ways.

If you follow Chelsey's advice you get $15 \cdot 210 = 3150$ ways. Her method creates a false labeling of the women into "initially chosen women" and "women chosen later"; this inflates the number of ways of accomplishing the task.

3. Use mathematical induction to prove that the sum of the first n odd positive integers is n^2 . Produce a formula with appropriate Σ -notation.

Solution. The theorem reads as follows

Theorem. *If n is a positive integer,*

$$\sum_{k=0}^{n-1} (2k+1) = n^2.$$

We now prove this via induction. If $n = 1$, both sides are 1 and we are done. Assume the theorem holds for n . Then

$$\begin{aligned}\sum_{k=1}^n (2k+1) &= 2n+1 + \sum_{k=1}^{n-1} (2k+1) \\ &= 2n+1 + n^2 \\ &= (n+1)^2;\end{aligned}$$

the induction carries.

Another acceptable sum is

$$\sum_{k=1}^n (2k-1) = n^2.$$

4. Two students are asked to determine the number of ways of arranging 6 Qs, 7 Rs, 3 Ss and 3 Ts in a "word" of length 19. Blevins completes this assignment and gets the result

$$\frac{19!}{3!3!6!7!}$$

and Wilson does the same thing and he gets

$$\binom{19}{3} \binom{16}{3} \binom{13}{7} \binom{6}{6}.$$

Explain how Blevins and Wilson arrived at their (seemingly) disparate solutions.

Solution. Blevins says we should label the Qs as Q_1, \dots, Q_6 , the Rs as R_1, \dots, R_6 , the Ss as S_1, S_2, S_3 and the Ts as T_1, T_2, T_3 . With the labels in place there are $19!$ ways to arrange the letters. However, there are $3!3!6!7!$ ways to label the letters. Dividing to de-label we get Blevins's result.

Wilson says there are 19 blanks to distribute the letters into. He begins by distributing the Ss; this can be done $\binom{19}{3}$ ways. Then he distributes the Ts; since 16 slots remain, there are $\binom{16}{3}$ ways to do this. Now distribute the Rs; there are $\binom{13}{7}$ ways to do that. Finally plunk the Ss into the remaining slots. This gives us Wilson's solution.

5. How many strings of length 10 can be made from the word MISSISSIPPI?

Solution. There are 1 M, 4 Ss, 4 Is and 2 Ps. We create four cases based on the omitted letter. If you omit the M, you have $\binom{10}{442}$ ways. If you omit the S, you have $\binom{10}{1342}$ ways. If you omit the I, you have $\binom{10}{1342}$ ways. If you omit the P, you have $\binom{10}{1441}$ ways. Now add them to get the total.

Part II, You have a choice! Pick 5!

6. A valet has keys for 10 cars and 10 owners waiting for a car. In how many ways can he distribute the keys so that exactly 4 owners get their keys back and so the rest get wrong keys?

Solution. Choose four lucky owners that get their keys back; this can be done $\binom{10}{4}$ ways. Now derange the rest; this can be done D_6 ways. The total number of ways is

$$\binom{10}{4} D_6.$$

7. A basketball tournament has 16 teams. How many possible ways are there to pair the teams off for the first round?

Solution. Choose 8 labeled teams; there are

$$\binom{16}{22222222} = \frac{16!}{2^8}$$

ways to do this. There are $8!$ ways to label the teams so we divide by $8!$ to de-label and get

$$\frac{16!}{2^8 8!}$$

ways.

You can also do this. Take a team and pair it off; there are 15 ways to do this and 14 teams remain. Now pair a second team off; there are 13 ways to do this. Keep going: total number of ways is

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15.$$

8. Show that if ten nonnegative integers has sum 101, there must be a trio of integers with the ten that has a sum of at least 31.

Solution. Warning Will Robinson! You will be disgusted! Let us denote the integers by x_1, x_2, \dots, x_{10} . We now form the sum

$$\sum_{\text{alltriples } i,j,k} (x_i + x_j + x_k).$$

Consider any x_i in the list of numbers. You can extend this to a trio any of $\binom{9}{2} = 36$ ways. Thus, each x_i is in 36 triples so each x_i gets summed 36 ways. We now know

$$\sum_{\text{alltriples } i,j,k} (x_i + x_j + x_k) = 36 * 101 = 3636.$$

This sum has $\binom{10}{3} = 120$ summands. Observe that $3636/120$ is 30 with a positive remainder, so at least one summand is at least 31.

Here is a second solution due in part to Wilson. For convenience here, we will denote the integers by x_0, x_1, \dots, x_9 . Form the sum

$$\sum_{k=0}^9 x_k + x_{k+1} + x_{k+2}$$

where we define $x_{10+k} = x_k$ as necessary. Each x_k gets summed exactly three times. Therefore, we know that

$$\sum_{k=0}^9 x_k + x_{k+1} + x_{k+2} = 303.$$

Since there are ten summands here, at least one must exceed 30. Therefore a consecutive sum of three elements must be 31, where we consider sums such as $x_9 + x_0 + x_1$ to be consecutive. Cool idea, Mr. Wilson!

9. Find the number of ways to make change of a \$50 bill with twenties, tens, fives and ones.

Suppose we have a number divisible by 5, say $5n$. We shall count the number of ways to change that amount of money with ones and fives. You can use $0-n$ fives, then fill in with ones. This gives us a total of $n + 1$ ways.

We can change the fifty with:

no twenties	no tens	11 ways
no twenties	one ten	9 ways
no twenties	two tens	7 ways
no twenties	three tens	5 ways
no twenties	four tens	3 ways
no twenties	five tens	1 way
one twenty	no tens	7 ways
one twenty	one ten	5 ways
one twenty	two tens	3 ways
one twenty	three tens	1 way
two twenties	no tens	3 ways
two twenties	one ten	1 way

This yields a total of 56 ways.

10. Find a formula for $\sum_{k=1}^n \frac{1}{k(k+1)}$. Use induction to verify what you have found.

Solution. Do some fiddling and you will find that

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

We will now check this with induction. The initialization is easy. Assuming the induction hypothesis we operate as follows

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \frac{1}{(n+1)(n+2)} + \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} \\ &= \frac{n+1}{n+2}. \end{aligned}$$

The induction carries.

11. A poll is taken of 600 S&Mers about three favourite restaurants: Cosmic, Elmo's and Randy's Pizza. Three students disliked all of these restaurants and 500 liked all three. 520 students liked Elmo's and Randy's, and the same number liked Cosmic and Randy's. Ten students only liked Cosmic, 12 only liked Elmo's and 5 only liked Randy's. How many students liked Cosmic and Elmo's?

Solution. Fiddle with a venn diagram and get 530.

12. A staircase has n stairs; denote by s_n the number of ways to ascend the stairway if you can go up 1, 2 or 3 steps at a time. Write a recursive formula for s_n . Can you write an explicit formula? Compute s_n for $1 \leq n \leq 10$.

Solution. There are three types of ascents: one begins with a single step and there are s_{n-1} of these, another begins with a double step, and there are s_{n-2} of these and there are ones that begin with a triple step and there are s_{n-3} of these. It is clear that $s_1 = 1$ and that $s_2 = 2$. To find s_3 realize that there are the plans: 1-1-1, 1-2, 2-1 and 3, for a total of 4. We know $s_3 = 4$. Our complete recursion is

$$\begin{cases} s_n = s_{n-1} + s_{n-2} + s_{n-3} & \text{if } n \geq 4 \\ 4 & \text{if } n = 3 \\ 2 & \text{if } n = 2 \\ 1 & \text{if } n = 1. \end{cases}$$

We can now grind out more as in this table

1	2	3	4	5	6	7	8	9	10
1	2	4	7	13	24	44	81	149	274

The explicit formula is

$$\sum_{k,l} \binom{n-k-2l}{k,l}.$$

Here is how we get this sum. There will be some number, say k of double steps and some number, say l of triple steps. If there are n stairs altogether, it will the stepper will take $n - k - 2l$ steps to ascend the stairs. There are

$$\binom{n-k-2l}{k,l}$$

ways to decide which steps are doubles and which are triples. To finish, sum over all possible k and l .